

SCROLLS AND HYPERBOLICITY

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ABSTRACT. Using degeneration to scrolls, we give an easy proof of non-existence of curves of low genera on general surfaces in \mathbb{P}^3 of degree $d \geq 5$. We show, along the same lines, boundedness of families of curves of small enough genera on general surfaces in \mathbb{P}^3 . We also show that there exist Kobayashi hyperbolic surfaces in \mathbb{P}^3 of degree $d = 7$ (a result so far unknown), and give a new construction of such surfaces of degree $d = 6$. Finally we provide some new lower bounds for geometric genera of surfaces lying on general hypersurfaces of degree $3d \geq 15$ in \mathbb{P}^4 .

CONTENTS

Introduction	1
1. Scrolls	2
1.1. Generalities on scrolls	2
1.2. Surface scrolls with ordinary singularities	3
1.3. Surface scrolls with general moduli	4
2. Bounding degrees of low genera curves on surfaces	7
2.1. Algebraic hyperbolicity	7
2.2. Bounding degrees of curves of low genera on general surfaces in \mathbb{P}^3	8
2.3. Families of low degree curves of a given genus on general surfaces in \mathbb{P}^3	9
3. Bounding geometric genera of divisors on general 3-folds in \mathbb{P}^4	11
4. Degeneration to scrolls and Kobayashi hyperbolicity	12
4.1. Limiting Brody curves and Hurwitz Theorem	12
4.2. A hyperbolicity criterion for hypersurfaces in \mathbb{P}^n	12
4.3. Applying scrolls to Kobayashi hyperbolicity	13
References	15

INTRODUCTION

What is the lowest geometric genus $\eta(n, d)$ of a reduced, irreducible curve on a very general hypersurface of degree d in \mathbb{P}^n ? The case $n = 2$ is trivial. For $n = 3$ one has

$$\eta(3, d) = 0 \quad \text{if } d \leq 4 \quad \text{while} \quad \eta(3, d) = \binom{d-1}{2} - 3 \quad \text{if } d \geq 5 \quad (1)$$

and for any $d \geq 6$ this bound is achieved by tritangent plane sections, and only by these [49]. Similarly, $\eta(4, d) = 0$ if $d \leq 5$, while $\eta(4, 6) \geq 2$ [14]. More generally, for $n \geq 4$ one has

$$\eta(n, d) = 0 \quad \forall d \leq 2n - 3 \quad \text{and} \quad \eta(n, d) \geq 1 \quad \forall d \geq 2n - 2$$

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see [46] (in the case $d = 2n - 3$ see also [26, 36, 45]). Presumably, $\eta(n, d) \rightarrow \infty$ as $d \rightarrow \infty$, however, the asymptotic of $\eta(n, d)$ is unknown. One is equally interested in bounds for the geometric genus or other numerical invariants of higher dimensional subvarieties in general hypersurfaces, see e.g. [12, 18, 19, 37, 47, 48, 50].

A projective variety X is *algebraically hyperbolic* if it does not admit a non-constant morphism from an abelian variety. If there is an algebraically hyperbolic hypersurface of degree d in \mathbb{P}^n , then a very general hypersurface of degree d is algebraically hyperbolic as well. For instance, a very general surface X of degree $d \geq 5$ in \mathbb{P}^3 is algebraically hyperbolic. Indeed, X does not contain rational or elliptic curves since $\eta(3, d) \geq 3$ for $d \geq 5$ by (1). This also follows from Proposition 2.1 below if $d \geq 6$, while Corollary 2.2 offers a short proof of Xu's and Voisin's result about non-existence of rational curves on a very general quintic in \mathbb{P}^3 . Since X is of general type it cannot be dominated by an abelian variety.

Similarly, a general sextic threefold X in \mathbb{P}^4 is algebraically hyperbolic. Indeed, X does not contain rational or elliptic curves since $\eta(4, 6) \geq 2$. By [50, Theorem 1] it also does not contain surfaces with desingularization of geometric genus at most 2. Therefore, every map from an abelian variety to X is constant.

A variety X is *Kobayashi hyperbolic*, or simply *hyperbolic*, if it does not admit any non-constant entire curve $\mathbb{C} \rightarrow X$. Hyperbolicity implies algebraic hyperbolicity, and it is stable under small deformations.

Given one of the two above hyperbolicity notions, one can ask what is the lowest degree $d = d(n)$ such that a very general projective hypersurface in \mathbb{P}^n of degree d possesses this property. For instance, the classical *Kobayashi problem* suggests that a very general hypersurface of degree $d \geq 2n - 1$ in \mathbb{P}^n is hyperbolic.

It is known that, indeed, a very general surface of degree $d \geq 18$ in \mathbb{P}^3 is hyperbolic [38] (see also [15, 31]). The existence of hyperbolic surfaces in \mathbb{P}^3 of degree d for all $d \geq 8$ was established with a degeneration argument in [44] (see the references in [44] for other constructions), and for $d = 6$ in [17]. In §4.3 below (see, in particular, Theorem 4.6) we give an alternative proof for the case $d = 6$, which works also in the (so far unknown) case $d = 7$. The case $d = 5$ in the Kobayashi problem for \mathbb{P}^3 remains open.

Our method consist in degenerating a general hypersurface to a certain special one, following the limits in the degeneration of entire curves or of algebraic curves or surfaces, according to the hyperbolicity notion we are dealing with. In this framework the concept of Brody curves and their limits is very useful, cf. e.g. [43, 44, 51, 52]. We recall a minimum of basics on this subject in §4. Our preferable degenerations here are to *scrolls*, and we recall their main properties in §1. In subsection 2.1 we give an easy proof of non-existence of curves of low genera on very general surfaces in \mathbb{P}^3 of a given degree. In §3 we treat the higher dimensional case. In particular, in Theorem 3.3 we provide a lower bound for the geometric genus of surfaces contained in very general hypersurfaces of degree $3d \geq 15$ in \mathbb{P}^4 .

By a well-known theorem of Bogomolov [8], on a smooth surface S of general type with $c_1^2(S) > c_2(S)$, the curves of a fixed geometric genus vary in a bounded family. This was partially extended in [30] to any smooth surface S of general type by showing that there are only a finite number of rational and elliptic curves on S with a fixed number of nodes and ordinary triple points and no other singularities. In subsection 2.2 we address the question whether curves of a given geometric genus have bounded degree on a general surface of degree $d \geq 5$ in \mathbb{P}^3 . We give an affirmative answer for all genera $g \leq d^2 + O(d)$.

Finally in subsection 4.3 we prove the aforementioned Theorem 4.6.

1. SCROLLS

1.1. Generalities on scrolls. By a *scroll* in \mathbb{P}^n we mean the image $\Sigma = \varphi(S)$ of a smooth, proper \mathbb{P}^1 -bundle $\pi : S \rightarrow E$ under a birational morphism $\varphi : S \rightarrow \Sigma \hookrightarrow \mathbb{P}^n$ which sends the *rulings* of S (i.e. the fibres of π) to projective lines, called *rulings* of Σ . The variety E is called the *base* of the scroll. We will denote by H and F a hyperplane section and a ruling of Σ , respectively. We may abuse notation denoting by H and F also their proper transforms on S .

The induced morphism $\mu : E \rightarrow \text{Gr}(1, n)$ to the Grassmanian of lines in \mathbb{P}^n is birational onto its image. Any such morphism μ appears in this way, where $\pi : S \rightarrow E$ is induced via μ by the tautological \mathbb{P}^1 -bundle

over the Grassmanian. Furthermore, $d = \deg(\Sigma)$ is equal to the degree of the subvariety $\mu(E)$ under the Plücker embedding of the Grassmanian [5, 12.4], [16, 11.4.1].

We will suppose from now on that $\varphi : S \rightarrow \Sigma$ coincides with the normalization morphism. We denote by $\text{br}(\Sigma)$ the set of *multibranch points* of Σ , i.e. the set of points $x \in \Sigma$ such that $\varphi^{-1}(x)$ consists of more than one point. If $x \notin \text{br}(\Sigma)$, e.g. x is a smooth point of Σ , then there is just one ruling passing through x . Since φ is finite, there is no point on Σ which belongs to infinitely many rulings.

We let $\Delta_\Sigma = \overline{\text{br}(\Sigma)} \subseteq \Sigma$ and $\Delta_S = \varphi^{-1}(\Delta_\Sigma) \subseteq S$. We will assume that the following conditions hold:

- (C1) $\dim(\Sigma) = n - 1$;
- (C2) Δ_Σ coincides with $\text{Sing}(\Sigma)$;
- (C3) Δ_Σ and Δ_S are both irreducible of dimension $n - 2$;
- (C4) a general point $x \in \Delta_\Sigma$ is a normal crossing double point of Σ . In particular, $\varphi^{-1}(x)$ has cardinality 2, and x sits on two different rulings;
- (C5) Δ_Σ contains no ruling, i.e. $\mu : E \rightarrow \text{Gr}(1, n)$ is injective.

In this situation Δ_Σ and Δ_S both have natural scheme structures, and Δ_S is a reduced divisor on S .

Conditions (C1)–(C4) are verified if $S \subseteq \mathbb{P}^{n+k}$ is a smooth scroll of dimension $n - 1$ and $\varphi : S \rightarrow \Sigma$ is induced by a general linear projection $\mathbb{P}^{n+k} \dashrightarrow \mathbb{P}^n$; see [23]. The last condition (C5) can be easily checked by induction; we leave the details to the reader.

Lemma 1.1. *In the above setting, a general ruling of Σ meets the double locus Δ_Σ in $d - n + 1$ points. In particular, Σ is swept out by an $(n - 2)$ -dimensional family of $(d - n + 1)$ -secant lines of Δ_Σ .*

Proof. By the Ramification Formula [24, 9.3.7(b)] there is a linear equivalence relation on S

$$\Delta_S \sim (d - n - 1)H - K_S. \quad (2)$$

Since $F \cdot H = 1$ and $F \cdot K_S = -2$, we have $F \cdot \Delta_S = d - n + 1$. Since Δ_S is reduced, the general ruling of S meets Δ_S in $d - n + 1$ distinct points. The assertions follow because φ induces an isomorphism of each ruling of S to its image. \square

1.2. Surface scrolls with ordinary singularities. We restrict here to the case $n = 3$. So E is a smooth curve of genus g and $S \subseteq \mathbb{P}^{3+k}$ and $\Sigma \subseteq \mathbb{P}^3$ are surfaces, called *scrolls of genus g* : here g is the *sectional genus* of the scroll.

Remark 1.2. For an irreducible curve C on S of genus g' such that $C \cdot F = \nu$, the Riemann–Hurwitz Formula implies the inequalities $g' \geq \nu(g - 1) + 1 \geq g$. In particular, for $g \geq 1$ the only irreducible curves on S of geometric genus $g' < g$ are the rulings, and for $g' = g \geq 2$ the curve C is a *unisecant* i.e., the intersection number ν of C with rulings is 1. The same holds on Σ .

We say that Σ has *ordinary singularities* if, in addition to conditions (C1)–(C5), the following hold:

- (C6) the singularities of the *double curve* Δ_Σ consist of finitely many triple points, which are also ordinary triple points of the surface Σ (these are locally analytically isomorphic to the surface singularity $xyz = 0$ in \mathbb{C}^3 at the origin);
- (C7) the non-normal crossings singularities of Σ are finitely many *pinch points*. These are the points in $\Delta_\Sigma \setminus \text{br}(\Sigma)$, and there is just one ruling through each of them. A pinch point has just one preimage on S , which, abusing terminology, we will also call a pinch point;
- (C8) the only singularities of Δ_S are ordinary double points, three of them over each triple point of Δ_Σ .

Furthermore, the degree two map $\varphi : \Delta_S \rightarrow \Delta_\Sigma$ is ramified exactly over the pinch points of Σ .

These are the singularities of a general projection to \mathbb{P}^3 of a smooth surface in \mathbb{P}^4 , or even of a surface in \mathbb{P}^4 with finitely many *nodes*, i.e. double points with tangent cone formed by two planes spanning \mathbb{P}^4 . In this case the curves Δ_Σ and Δ_S are irreducible, except for the projection in \mathbb{P}^3 of the Veronese surface of degree 4 in \mathbb{P}^5 (cf. [21, 22, 32, 33]). Note that a general projection to \mathbb{P}^4 of any smooth surface in \mathbb{P}^r (with $r > 4$) has only nodes as singularities and the Veronese surface of degree 4 in \mathbb{P}^5 is the only one whose general projection to \mathbb{P}^4 is smooth (see [41, 53]).

The basic invariants of S are

$$c_1^2 = K_S^2 = 8(1 - g), \quad c_2 = e(S) = 4(1 - g), \quad \text{and} \quad \chi(\mathcal{O}_S) = \frac{c_1^2 + c_2}{12} = 1 - g$$

(see [20], [26, Ch. 5, §2]). The following *projective invariants* are also important

$$\begin{aligned} \delta_\Sigma &= \deg(\Delta_\Sigma) \\ \gamma_\Sigma &= \text{the geometric genus of } \Delta_\Sigma \\ t_\Sigma &= \text{the number of triple points of } \Delta_\Sigma \\ p_\Sigma &= \text{the number of pinch points of } \Sigma \\ \tilde{\gamma}_\Sigma &= \text{the geometric genus of } \Delta_S \end{aligned} \tag{3}$$

(in the sequel we suppress the index Σ when unnecessary).

For the proof of the following formulas see e.g. [8], [16, §11.5], [20, p. 176], [39], [42, (1)-(10)], and references therein.

Proposition 1.3. *Let Σ stands as before for a scroll in \mathbb{P}^3 of degree d and genus g with ordinary singularities. Then the projective invariants of Σ are given by the Bonnesen's formulas*

$$\delta = \binom{d-1}{2} - g, \tag{4}$$

$$\gamma = \binom{d-3}{2} + (d-5)g, \tag{5}$$

$$t = \binom{d-2}{3} - (d-4)g, \tag{6}$$

$$p = 2d + 4(g-1), \tag{7}$$

$$\tilde{\gamma} = 2(\gamma + g) + d - 3. \tag{8}$$

Remark 1.4. Due to (6), for $d \geq 5$ the inequality $t \geq 0$ reads $g \leq \frac{1}{6}(d-2)(d-3)$. This implies

$$g \leq d-4, \quad \text{if } d = 5, 6, 7. \tag{9}$$

In the sequel we also need the inequality

$$\gamma > 3(g-1) \quad \text{for all } g \geq 1 \text{ and } d \geq 5. \tag{10}$$

This follows from (5) for $d \geq 8$ and from (5) and (9) for $d = 5, 6, 7$ (actually, $\gamma > 3g$ for all $g \geq 1$ and $d \geq 5$ except for $g = 2, d = 6$).

1.3. Surface scrolls with general moduli. We recall a result from [4] (cf. also [10, Theorem 1.2]).

Theorem 1.5. *Let $g \geq 0$ be an integer and let $k = \min\{1, g-1\}$. If $d \geq 2g+3+k$, then there exists a unique irreducible component $\mathcal{H}_{d,g}$ of the Hilbert scheme of scrolls of degree d and sectional genus g in \mathbb{P}^r , where $r = d - 2g + 1$, such that the general point $[S] \in \mathcal{H}_{d,g}$ represents a smooth scroll S with $h^1(S, \mathcal{O}_S(1)) = 0$, i.e. S is non-special. Furthermore $\mathcal{H}_{d,g}$ dominates the moduli space \mathcal{M}_g of smooth curves of genus g via the map sending a scroll to its base.*

Remarks 1.6. (i) Assuming that $d \geq 2g + 3 + k$ (as in the above theorem), we have $r \geq 3$ if $g = 0$, $r \geq 4$ if $g = 1$, and $r \geq 5$ if $g \geq 2$, and we can project smooth scrolls S with $[S] \in \mathcal{H}_{d,g}$ thus obtaining scrolls Σ in \mathbb{P}^3 with ordinary singularities and irreducible double curve.

(ii) The assumption of Theorem 1.5 gives $d \geq 2g + 4$ for $g \geq 2$ and $d \geq 2g + 3 = 5$ for $g = 1$. In fact, similar results hold also for $g \geq 2$ and $d = 2g + 3$ or $d = 2g + 2$, while the corresponding scrolls are no longer smooth.

More precisely, let $g \geq 2$ and $d = 2g + 3$ (i.e., $r = 4$). Then $\mathcal{H}_{d,g}$ is a component of the Hilbert scheme, whose general point $[S'] \in \mathcal{H}_{d,g}$ represents a scroll $S' \subseteq \mathbb{P}^4$ with only nodes as singularities and with a smooth normalization S such that $h^0(S, \mathcal{O}_S(1)) = 5$ and $h^1(S, \mathcal{O}_S(1)) = 0$ (this can be shown with the same analysis as in [10]). Once again, $\mathcal{H}_{d,g}$ dominates the moduli space \mathcal{M}_g .

If $g \geq 2$ and $d = 2g + 2$ (i.e., $r = 3$), a similar assertion holds. However, now $\mathcal{H}_{d,g}$ is no longer a component of the Hilbert scheme, but a locally closed subset of the projective space $\mathcal{L}_d = |\mathcal{O}_{\mathbb{P}^3}(d)|$ of all surfaces of degree d in \mathbb{P}^3 . It is reasonable to expect that a general point $[\Sigma] \in \mathcal{H}_{d,g}$ represents a scroll $\Sigma \subseteq \mathbb{P}^3$ with ordinary singularities. This would follow by going deeper into the analysis performed in [10], but we do not use this here in the full generality. We investigate below in more detail various examples (see especially Example 1.9).

Example 1.7. Elliptic quartic scrolls. Let E be a smooth curve of type (a, b) on $\mathbb{P}^1 \times \mathbb{P}^1$, identified with a smooth quadric in \mathbb{P}^3 . The genus of E is $g = ab - a - b + 1$. Consider a pair of skew lines R_1, R_2 in \mathbb{P}^3 . Identifying these lines with the factors of $\mathbb{P}^1 \times \mathbb{P}^1$, we can interpret the canonical projections of E to the factors as maps $\varphi_i : E \rightarrow R_i$, $i = 1, 2$, of degree a and b , respectively. For each $x \in E$ we consider the line L_x joining the points $\varphi_i(x)$, $i = 1, 2$. This yields the map $\mu : x \in E \rightarrow L_x \in \text{Gr}(1, 3)$. Its image is a smooth curve on $\text{Gr}(1, 3)$ under the Plücker embedding of the Grassmanian $\text{Gr}(1, 3)$ as a quadric in \mathbb{P}^5 . The associated scroll

$$\Sigma = \Sigma_{a,b} = \bigcup_{x \in E} L_x$$

in \mathbb{P}^3 with base E has degree $a + b$. Indeed, it has singularities of multiplicities a along R_1 and b along R_2 . So a line $\langle A, B \rangle$, where $A \in R_1$ and $B \in R_2$, meets Σ only in A and B .

In particular, for $a = b = 2$ we obtain a quartic scroll in \mathbb{P}^3 of genus 1 with two skew double lines, and for $a = 3$, $b = 2$ a quintic scroll of genus 2 with a double line and a triple line.

From now on, we concentrate on an elliptic quartic scroll $\Sigma = \Sigma_{2,2}$. The preimage Δ_S of Δ_Σ on S consists of two disjoint copies E_1, E_2 of E with $\varphi_i : E_i \rightarrow R_i$, $i = 1, 2$, corresponding to two distinct g_2^1 's on E . There are in total 8 pinch points of Σ , 4 on each of the lines R_1, R_2 . These are the branch points of the maps φ_i , $i = 1, 2$. If these maps are sufficiently general, also the pinch points are generically located along R_1, R_2 and the ruling passing through a pinch point does not contain any other pinch point.

Let us illustrate on this example our degeneration method. Any smooth elliptic quartic curve is a complete intersection of two quadrics in \mathbb{P}^3 . Hence it embeds as well to the Grassmanian $\text{Gr}(1, 3)$. By virtue of Remark 1.6 to Theorem 1.5 (the case $r = 3$) these curves fill in a unique irreducible component $\mathcal{H}_{4,1}$ of the Hilbert scheme of curves of degree 4 in $\text{Gr}(1, 3)$, which dominates the moduli space \mathcal{M}_1 . The component $\mathcal{H}_{4,1}$ contains all *limit curves*, e.g. all reduced, nodal curves of degree 4 and arithmetic genus 1 spanning a \mathbb{P}^3 . For instance, the union E_0 of two conics Γ_1, Γ_2 meeting transversally at two distinct points f_1, f_2 is such a limit curve. The curve E_0 corresponds to the union Σ_0 of two quadrics surfaces Q_1, Q_2 in \mathbb{P}^3 associated to the conics Γ_1, Γ_2 on the Grassmanian $\text{Gr}(1, 3)$. We may assume these quadrics to be smooth. They intersect along the quadrilateral $F_1 \cup F_2 \cup G_1 \cup G_2$, where the lines F_1, F_2 correspond to f_1, f_2 and belong to the same ruling on each quadric, and G_1, G_2 are distinct lines belonging to the other ruling. We let $p_{ij} = F_i \cap G_j$, $i, j = 1, 2$.

The surface Σ_0 can be seen as a flat limit of surfaces of type Σ , since it corresponds to a point in $\mathcal{H}_{4,1}$. The limit of the ruling of Σ is the union of the two rulings of Q_1 and Q_2 containing F_1, F_2 . The limits of the double lines R_1, R_2 are the lines G_1, G_2 . The limit of each of the components E_i of the curve Δ_S on S consists of two copies of G_i glued at $p_{1i}, p_{2,i}$. Each of these points is the limit of two pinch points of Σ .

Conversely, when we deform Σ_0 to Σ , the two double lines F_1 and F_2 of Σ_0 disappear, because we are smoothing the two nodes of E_0 . Each of the points p_{ij} ($i, j = 1, 2$) gives rise to two pinch points generically located along the double line of Σ , which deforms G_j .

Example 1.8. *Elliptic quintic scrolls.* Consider now the case where $d = 5$ and $g = 1$. By Theorem 1.5, a general point $[S] \in \mathcal{H}_{5,1}$ represents a smooth scroll S in \mathbb{P}^4 , whose general projection Σ to \mathbb{P}^3 has ordinary singularities. According to Bonnesen's formulas (4)-(8), the double curve $C = \Delta_\Sigma$ is an irreducible, smooth, elliptic quintic curve, which contains the 10 pinch points of S . Its preimage $\tilde{C} = \Delta_S$ is a smooth, irreducible curve on S of genus 6. By Lemma 1.1, the rulings of Σ are trisecant lines to C .

Conversely, for any smooth elliptic quintic curve C in \mathbb{P}^3 , the trisecant lines to C sweep out a quintic scroll Σ , which is singular exactly along C (cf. Berzolari's Formula, Proposition 1 and Corollary 2 in [6]). Such a surface Σ is an elliptic scroll, and by the Riemann–Roch Theorem it comes as a projection of a surface represented by a point in $\mathcal{H}_{5,1}$ as above.

Any such scroll Σ corresponds to an embedding of an elliptic quintic curve E in $\text{Gr}(1, 3)$ via the map μ as in §1.1. The image of E is a quintic elliptic normal curve, contained in a hyperplane section of $\text{Gr}(1, 3)$. Indeed, any normal, elliptic quintic curve lies on some smooth quadric in \mathbb{P}^4 , hence on a hyperplane section of $\text{Gr}(1, 3)$.

There is another interpretation of these elliptic quintic scrolls. Let E be an elliptic curve. Consider its symmetric product $E(2)$, formed by all degree 2 effective divisors on E . The class of the *diagonal* $D = \{2p, p \in E\}$ is divisible by 2 in $\text{Pic}(E(2))$; we denote by ϑ the class of its half. One has $K_{E(2)} \sim -\vartheta$.

The Abel–Jacobi map $\alpha : E(2) \rightarrow \text{Pic}^{(2)}(E) \cong E$ makes $E(2)$ a \mathbb{P}^1 -bundle with base E . The rulings are the g_2^1 's on E . The *coordinate curves* $E_p = \{x + p, x \in E\} \cong E$ are unisecant curves of the rulings and form a one-dimensional family parametrized by the point p varying on E . We have $E_p^2 = 1$. If F_1, F_2 are rulings, then the divisor class of the curve $E_p + F_1 + F_2$ is very ample on $E(2)$ and maps isomorphically the surface $E(2)$ onto a quintic scroll S in \mathbb{P}^4 . Each coordinate curve E_p is mapped to a smooth plane cubic on S which is the residual intersection of S with a hyperplane containing two rulings. Conversely any smooth plane cubic on S is a coordinate curve: indeed, it sits on a 1-dimensional family of hyperplane sections of S and their residual intersections with S is a pair of lines.

Let as before Σ denote the image of S under a general projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$. Any coordinate curve on S is isomorphically mapped to a smooth plane cubic and the images on Σ of two distinct coordinate cubics on S are distinct. This provides a complete, one-parameter family of smooth plane cubic curves on Σ which are the only plane cubics on Σ . Let L be the plane containing one of them \bar{E} . The residual intersection on $L \cap \Sigma$ must be a union of two rulings, which meet on C . The corresponding rulings on S span a hyperplane which cuts out on S a coordinate cubic \tilde{E} plus the two rulings. Hence \bar{E} is the image of \tilde{E} on Σ .

Let $x \in C$ be a general point and F_1, F_2 the two rulings through x . The plane π spanned by them cuts Σ in the union of F_1, F_2 and a smooth cubic \bar{E} , which is the projection of a unique coordinate curve. When x varies, we obtain in this way all projections of coordinate curves. This shows that C is isomorphic to E , since it parametrizes the family of coordinate curves.

When the center of projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ varies we obtain a monodromy action. The following argument shows that this monodromy is irreducible on appropriately chosen objects.

The cubic curve \bar{E} as above does not pass through x , and cuts the ruling F_i in three (generically distinct) points p_i, q_{i1}, q_{i2} , $i = 1, 2$, such that $q_{i1}, q_{i2} \in C$. Indeed, p, q_{i1}, q_{i2} , $i = 1, 2$, are the five intersection points of π with C . By moving the centre of projection, we may assume that the pair of rulings (F_1, F_2) corresponds to a general divisor of a given g_2^1 on E , and that $q_{11} + q_{12}$ ($q_{21} + q_{22}$, respectively) is a general divisor in the g_2^1 cut out on \bar{E} by the lines through p_1 (through p_2 , respectively). In conclusion, by moving the centre of projection the monodromy interchanges the pairs $q_{11} + q_{12}$ and $q_{21} + q_{22}$ and also interchanges the points in each pair separately.

Example 1.9. *Sextic scroll of genus two.* By the case $g \geq 2$, $r = 3$ of Remark 1.6.2, there exist sextic scrolls Σ of genus two in \mathbb{P}^3 . They correspond to genus 2 curves of degree 6 on the Grassmanian $\text{Gr}(1, 3)$. In fact, by the Riemann–Roch Theorem, any smooth curve of genus 2 embeds in \mathbb{P}^4 as a sextic. This sextic

spans \mathbb{P}^4 and lies on a smooth quadric in \mathbb{P}^4 , hence on a hyperplane section of the Grassmanian $\mathrm{Gr}(1, 3)$. These curves fill in a unique component $\mathcal{H}_{6,2}$ of the Hilbert scheme of curves of degree 6 and genus 2 in $\mathrm{Gr}(1, 3)$, which dominates \mathcal{M}_2 via the natural map. As in Example 1.7.2, $\mathcal{H}_{6,2}$ contains limit curves, and in particular all reduced, nodal curves of degree 6 and arithmetic genus 2 spanning a \mathbb{P}^4 .

Assuming that a general such scroll Σ has ordinary singularities, Lemma 1.1 and Proposition 1.3 say that the rulings of Σ are four-secant lines to the double curve $C = \Delta_\Sigma$, which is a smooth, irreducible curve in \mathbb{P}^3 of degree 8 and genus 5, passing through all 16 pinch points of Σ . The preimage $\tilde{C} = \Delta_S$ of C on S is a smooth curve of degree 16 and of genus 17.

Let us show that a general sextic scroll Σ in \mathbb{P}^3 of genus 2 has ordinary singularities and an irreducible double curve C . Consider a reducible sextic curve $E_0 \subseteq \mathbb{P}^4$ of arithmetic genus 2, which consists of a general smooth elliptic normal quintic curve E' and a line D meeting E' transversally in two distinct points. Such a curve E_0 corresponds to a point in $\mathcal{H}_{6,2}$, hence to a reducible surface Σ_0 , which is a limit of genus 2 sextic scrolls Σ . On the other hand, Σ_0 is the union of a general quintic elliptic scroll Σ' in \mathbb{P}^3 arising from E' as in Example 1.8 plus a plane π through the two rulings F_1, F_2 of Σ_0 , which correspond to the intersection points of E' and D . These rulings meet at a point p of the double curve C' of Σ' . The ruling on π is given by the pencil of lines passing through p , which corresponds to the line D . The plane π cuts out on Σ' the union of the rulings F_1, F_2 and a smooth plane cubic \bar{E} , as described in Example 1.8. The singularities of Σ' consist of C' , F_1, F_2 , and \bar{E} .

When we deform E_0 to a general smooth sextic E on $\mathrm{Gr}(1, 3)$, the scroll Σ_0 is deformed to an irreducible sextic scroll Σ . The double lines F_1 and F_2 of Σ_0 disappear, because we are smoothing the two nodes of E_0 . This means that the flat limit on Σ_0 of the singular locus of Σ is the nodal curve $C_0 = C' \cup \bar{E}$ of arithmetic genus 5. Hence Σ is singular only along a double curve C , which has arithmetic genus 5. The latter curve is irreducible. Indeed, otherwise this would be still a union of the form $C' \cup \bar{E}$, and so the four-secant lines to C would sweep out a union of an elliptic scroll and a plane. However, this is impossible since Σ is irreducible and swept out by the four-secants of the double curve $C = \Delta_\Sigma$.

Since C_0 is nodal so is C . We claim that C is actually smooth. Indeed, we may restrict our family to a general irreducible curve germ in $\mathcal{H}_{2,6}$ through Σ_0 , and then normalize this germ. In this way we obtain a family of sextic scrolls over the disc \mathbb{D} with a family $\mathcal{C} \rightarrow \mathbb{D}$ of double curves. The central fibre of \mathcal{C} is a reducible nodal curve $C_0 = C' \cup \bar{E}$ with 4 nodes. Assuming that no one of these nodes is smoothed on a general fibre C of the family, C should also have 4 nodes. These nodes represent an étale four-sheeted cover over the disc. Now we can normalize the fibres of the family \mathcal{C} simultaneously (see e.g., [40]), thus obtaining a smooth family with an irreducible general fibre and a disconnected central fibre. The latter contradicts the *Connectedness Principle* (see [27, Ch. III, Ex. 11.4, p. 281]).

Consequently, at least one of the four nodes of C_0 has to be smoothed in the deformation to C . But then by the irreducibility of the monodromy (see the final part of Example 1.8) all nodes of C_0 have to be smoothed.

2. BOUNDING DEGREES OF LOW GENERA CURVES ON SURFACES

2.1. Algebraic hyperbolicity. Scrolls can be used to establish algebraic hyperbolicity of very general surfaces of a given degree d in \mathbb{P}^3 . For $d \geq 6$ this is done in Proposition 2.1 below. In the proof we use the Albanese inequality (see [1, 35] (see also [34, §4(b)]), which says the following: if a reduced projective curve C of geometric genus g degenerates into an effective cycle $C_0 = \sum_i m_i C_i$, where C_i is a reduced projective curve of geometric genus g_i , then

$$g \geq \sum_{g_i \geq 1} (m_i(g_i - 1) + 1). \quad (11)$$

In particular, $m_i(g_i - 1) \leq g - 1$ if $g_i \geq 1$. So $g_i \leq g$ for all i .

Proposition 2.1. *Assume that there exists a scroll Σ of degree $d \geq 5$ and genus $g \geq 1$ in \mathbb{P}^3 with ordinary singularities. Then a very general surface X in \mathbb{P}^3 of degree d does not contain curves of geometric genus $g' < g$.*

Proof. Let X be a very general surface in \mathbb{P}^3 of degree d . By the Noether-Lefschetz Theorem, the Picard group of X is generated by $\mathcal{O}_X(1)$. Consider the pencil $\{X_t\}_{t \in \mathbb{P}^1}$ generated by $X_0 = \Sigma$ and $X_\infty = X$. This gives rise to a flat family of surfaces $f : \mathcal{X} \rightarrow \mathbb{D}$ over a disc \mathbb{D} , where the central fibre over 0 is X_0 , all fibres X_t with $t \in \mathbb{D} \setminus \{0\}$ are smooth and $\text{Pic}(X_t)$ is generated by $\mathcal{O}_{X_t}(1)$ for a very general such fibre. We claim that a very general surface of this family does not contain any curve of geometric genus $g' < g$. We argue by contradiction and assume that this is not the case for some $g' < g$.

For each positive integer n we may consider the locally closed subset $\mathcal{H}_{n,g'}$ of the relative Hilbert scheme of $f : \mathcal{X} \setminus X_0 \rightarrow \mathbb{D} \setminus \{0\}$, whose points correspond, for each $t \neq 0$, to the irreducible curves of geometric genus g' in $|\mathcal{O}_{X_t}(n)|$. By our assumption, there is a component of $\mathcal{H}_{n,g'}$ which dominates $\mathbb{D} \setminus \{0\}$. Let \mathcal{H} be the closure of this component in the relative Hilbert scheme of $f : \mathcal{X} \rightarrow \mathbb{D}$. By the properness of the relative Hilbert scheme, \mathcal{H} surjects onto \mathbb{D} . Hence there is a curve $C_0 \in \mathcal{O}_{X_0}(n)$ on X_0 , which corresponds to a point in \mathcal{H} . By Albanese's inequality (11), every component of C_0 has geometric genus $g'' \leq g' < g$. By (5) (for $g \geq 2$) and Example 1.8 (for $g = 1$) we have $\gamma \geq g > g''$, where γ stands as before for the geometric genus of the double curve Δ_Σ of $X_0 = \Sigma$. Hence no component of C_0 coincides with Δ_Σ . Now the pull-back Γ of C_0 on the normalization $\varphi : S \rightarrow \Sigma$ belongs to the linear system $|\varphi^*(\mathcal{O}_\Sigma(n))|$ and maps birationally to C_0 by the finite map φ . Since the only curves of genus smaller than g on S are rulings, Γ consists of rulings. In particular, $\Gamma^2 = 0$. On the other hand, since $\Gamma \in |\varphi^*(\mathcal{O}_\Sigma(n))| = |\mathcal{O}_S(n)|$ we have $\Gamma^2 = n^2d > 0$, a contradiction. \square

In Proposition 2.10 below we slightly strengthen Proposition 2.1, using Proposition 2.8 and Corollary 2.9.

Keeping in mind Example 1.8, Proposition 2.1 provides an alternative quick proof of the following result originally established by Xu [49] and Voisin [46, 47].

Corollary 2.2. *On a very general surface of degree $d \geq 5$ in \mathbb{P}^3 there is no rational curve.*

Very general in Corollary 2.2 can be replaced by *general* provided the following question is answered in negative.

Question 2.3. *Does there exist a sequence of smooth quintic surfaces X_n in \mathbb{P}^3 such that X_n contains a rational curve of degree d_n and not smaller, with $d_n \rightarrow \infty$?*

Remark 2.4. Notice that for any integers $n \geq 3$, $d > 0$ and $0 \leq \delta \leq d^2(n-1) + 1$, the linear system $|\mathcal{O}_S(d)|$ on a general K3 surface S of degree $2n-2$ in \mathbb{P}^n with Picard group generated by $\mathcal{O}_S(1)$, contains a $(d^2(n-1) - \delta + 1)$ -dimensional family of irreducible δ -nodal curves, whose geometric genus equals $d^2(n-1) - \delta + 1$ (see [11]). So S contains nodal curves of every geometric genus $g \geq 0$. This applies in particular to general quartic surfaces in \mathbb{P}^3 .

2.2. Bounding degrees of curves of low genera on general surfaces in \mathbb{P}^3 . In this section we address the following *boundedness question* (cf. [8, 30] and the related discussion in the Introduction):

Question 2.5. *Given integers $d \geq 5$ and $g \geq 0$, does there exist a bound $n_{d,g}$ such that every irreducible curve of geometric genus g on a very general surface of degree d in \mathbb{P}^3 has degree $n \leq n_{d,g}$?*

If $d = 4$ the answer is negative (see [11, 25] and Remark 2.4). The argument in the proof of Propositions 2.1 and 2.10 can be used to give an affirmative answer for $d \geq 6$ and small enough g .

Proposition 2.6. *Suppose there exists a scroll Σ of degree $d \geq 6$ and genus $g \geq 2$ with ordinary singularities. Then the answer to Question 2.5 is affirmative for all genera $g' < \gamma$, where γ is defined in (3).*

Proof. We apply the same argument as in the proof of Proposition 2.1. Keeping the notation of this proposition, we let again $C_0 \in |\mathcal{O}_\Sigma(n)|$ denote a curve which is a limit of a flat family of irreducible curves $\{C_t\}_{t \in \mathbb{D} \setminus \{0\}}$, $C_t \in |\mathcal{O}_{X_t}(n)|$, of genus g' , where $g' \geq g \geq 2$ by Proposition 2.1. Write $C_0 = m_1 C_1 + \dots + m_h C_h + C'$ as a cycle, where for every $i = 1, \dots, h$ the curve C_i is irreducible of geometric genus $g_i \geq 1$ and its transform on S has positive intersections n_i with the rulings, whereas C' consists of rulings. Note that

$n = \sum_{i=1}^h m_i n_i$. By Albanese's inequality (11) and our hypothesis $g' < \gamma$, none of the components of C_0 coincides with Δ_Σ , and

$$g' \geq h + \sum_{i=1}^h m_i (g_i - 1).$$

The Riemann–Hurwitz formula yields: $g_i - 1 \geq n_i(g - 1)$ for all $i = 1, \dots, h$, so that $\gamma > g' \geq h + n(g - 1)$. This provides a bound $n < (\gamma - 1)/(g - 1)$ (we remind that $g \geq 2$). \square

Corollary 2.7. *Question 2.5 has an affirmative answer for*

$$\begin{aligned} d = 6, \quad g &\leq 5, \\ d \geq 7 \quad \text{even}, \quad g &< (d - 4)^2, \\ d \geq 7 \quad \text{odd}, \quad g &< \frac{(d - 3)(2d - 9)}{2}. \end{aligned}$$

Proof. For $d = 6$ we use the sextic scroll of genus 2 as in Example 1.9. For $d \geq 7$ even we write $d = 2m + 4$ and we consider in \mathbb{P}^3 general projections of smooth scrolls of genus m and degree d in \mathbb{P}^5 as in Theorem 1.5. For $d \geq 7$ odd we write $d = 2m + 3$ and we consider general projections of scrolls of genus m and degree d in \mathbb{P}^4 as in Remark 1.6.2. Applying Proposition 2.6 and taking into account (5), the assertion follows. \square

2.3. Families of low degree curves of a given genus on general surfaces in \mathbb{P}^3 . Proposition 2.8 below extends a similar result by Arbarello–Cornalba [2, Theorem 3.1], [3] and Zariski [54]; cf. also Knutsen [28, Lemma 4.4].

Let S be a smooth projective surface, $\text{Hilb}_1(S)$ the Hilbert scheme of curves on S , and $\mathcal{V}_g(S)$ the locally closed subset of $\text{Hilb}_1(S)$ formed by irreducible curves of geometric genus g .

Proposition 2.8. *In the setting as before, for an irreducible component \mathcal{V} of $\mathcal{V}_g(S)$ we let $v = \dim(\mathcal{V})$ and $\kappa = K_S \cdot \Gamma$, where a curve Γ on S corresponds to a general point in \mathcal{V} . Then $v \leq \max\{g, g - 1 - \kappa\}$. Furthermore, if $v > g$ then $v = g - 1 - \kappa$, and the general curve Γ of \mathcal{V} has only nodes as singularities.*

Proof. Let $f : C \rightarrow \Gamma$ be the normalization. The exact sequence

$$0 \rightarrow T_C \rightarrow f^*(T_S) \rightarrow N_f \rightarrow 0$$

defines the *normal sheaf* N_f to the map $f : C \rightarrow S$. It can be included into an exact sequence

$$0 \rightarrow \tau \rightarrow N_f \rightarrow N' \rightarrow 0,$$

where τ is the torsion subsheaf of N_f supported at the points, where the rank of the differential of f drops, and N' is an invertible sheaf. Due to the *Horikawa inclusion* $T_{[\Gamma]}(\mathcal{V}) \subseteq H^0(C, N')$ (see [2, (1.3)] or [3, Lemma 1.4]) we have $v \leq h^0(C, N')$. By Riemann–Roch,

$$h^0(C, N') = \deg(N') - g + 1 + h^1(C, N'), \quad \text{where} \quad \deg(N') \leq \deg(N_f) = 2g - 2 - \kappa.$$

If $h^1(C, N') = 0$ this gives $v \leq g - 1 - \kappa$. Otherwise N' is special, so $h^0(C, N') \leq g$. In any case, $v \leq \max\{g, g - 1 - \kappa\}$, as stated.

If $v > g$ then $h^1(C, N') = 0$. Since $H^1(C, \tau) = 0$ this yields $H^1(C, N_f) = 0$. As in [2, proof of (1.5) and p. 96] this implies $\tau = 0$, hence Γ is immersed (i.e., has no cuspidal singularities). One ends the proof as in [2, pp. 96–98]. \square

For $\mathcal{L}_d = |\mathcal{O}_{\mathbb{P}^3}(d)|$ we let

$$N_d = \dim(\mathcal{L}_d) = \binom{d+3}{3} - 1. \tag{12}$$

Given a smooth surface X of degree d in \mathbb{P}^3 and non-negative integers n, g , we let $\mathcal{V}_{n,g} = \mathcal{V}_{n,g}(X)$ denote the locally closed subset of $\mathcal{L}_{X,n} = |\mathcal{O}_X(n)|$ formed by irreducible curves on X of geometric genus g . We also let

$$g_{d,n} = \frac{dn(d+n-4)}{2} + 1$$

denote the arithmetic genus of the curves in $\mathcal{L}_{X,n}$. Notice that $g_{d,n} = g + \nu$ if a general member of $\mathcal{V}_{n,g}$ is nodal with ν nodes.

Corollary 2.9. *Let X be a general surface of degree $d \geq 3$ in \mathbb{P}^3 . If $g \geq 0$ and $n \in \{1, 2\}$ are such that $\mathcal{V}_{n,g}$ is nonempty, then*

$$g_{d,1} - 3 \leq g \leq g_{d,1} \quad \text{if } n = 1 \quad \text{and} \quad g_{d,2} - 9 \leq g \leq g_{d,2} \quad \text{if } n = 2.$$

Furthermore, for every irreducible component \mathcal{V} of $\mathcal{V}_{n,g}$, its general curve has exactly ν nodes as singularities and its dimension is

$$3 - \nu = g - g_{d,1} + 3 \quad \text{if } n = 1 \quad \text{and} \quad 9 - \nu = g - g_{d,2} + 9 \quad \text{if } n = 2. \quad (13)$$

Proof. Let us show the assertion in the case $n = 2$, the case $n = 1$ being similar. Consider the incidence relation $I \subseteq \mathcal{L}_d \times \mathcal{L}_2$ consisting of all pairs (X, Q) such that X is smooth and Q and X intersect in an irreducible curve C of geometric genus g . Then I is locally closed and comes equipped with the natural projections $p : I \rightarrow \mathcal{L}_d$ and $q : I \rightarrow \mathcal{L}_2$.

Note that if $(X, Q) \in I$ and C is the intersection of X and Q , then we have a family of dimension $\dim(\mathcal{L}_{d-2}) + 1$ of pairs $(X', Q) \in I$ such that intersection of X' and Q is C : indeed we can take X' general in the span of X and of all surfaces of degree d containing Q .

By our assumption p is dominant. Let I' be an irreducible component of I which dominates \mathcal{L}_d via p , so that $\dim(I') \geq N_d$. We assume that $q(I')$ contains a smooth quadric Q (the argument is similar otherwise, the details are left to the reader). Then I' dominates \mathcal{L}_2 via q and we may assume Q to be a general quadric. All components of $q^{-1}(Q)$ have dimension $\dim(I') - \dim(\mathcal{L}_2)$. Any such component can be identified with a family of surfaces of degree d . By the above discussion, the family of curves \mathcal{V} they cut out on Q has dimension

$$v = \dim(I') - \dim(\mathcal{L}_{d-2}) - \dim(\mathcal{L}_2) - 1.$$

Moreover, \mathcal{V} is an irreducible component of $\mathcal{V}_{d,g}(Q)$. We have

$$v \geq N_d - N_{d-2} - N_2 - 1 = g_{d,2} + 4d - 10 > g_{d,2} \geq g.$$

By Proposition 2.8, one has $v = g - 1 + 4d$, which yields $g_{d,2} - g \leq 9$. Furthermore, by Proposition 2.8 the general curve in \mathcal{V} has at most nodes as singularities, which implies (13). \square

Corollary 2.9 could be extended to handle also the case $n = 3$. This requires however to analyze a number of cases, which we avoid here.

Now we can strengthen Proposition 2.1 as follows.

Proposition 2.10. *Assume that there exists a scroll Σ of degree $d \geq 5$ and genus $g \geq 1$ in \mathbb{P}^3 with ordinary singularities. Then a very general surface X of degree d in \mathbb{P}^3 does not contain curves of geometric genus $g' \leq 3(g - 1)$.*

Proof. By Proposition 2.1 we may suppose that $d \geq 6$ and $g' \geq g \geq 2$. We proceed as in the proof of this proposition, using the same notation. We argue by contradiction and assume that there is a positive $g' \leq 3(g - 1)$, a positive integer n and a component of $\mathcal{H}_{n,g'}$ which dominates $\mathbb{D} \setminus \{0\}$. Consider a curve $C_0 \in \mathcal{O}_\Sigma(n)$ as in the proof of Proposition 2.1. As shown in this proof, C_0 cannot be composed of rulings. Hence it contains a component C_i of geometric genus $g_i > 0$, appearing in C_0 with multiplicity m_i . By Albanese's inequality (11) one has $g' - 1 \geq m_i(g_i - 1)$. By (10) and our assumption $g' \leq 3(g - 1) < \gamma$, hence $C_i \neq \Delta_\Sigma$. Therefore C_i lifts birationally to the normalization S of Σ yielding a ν_i -secant of the ruling on S . Combining the inequalities above, by Hurwitz Formula (see Remark 1.2) we obtain

$$3(g - 1) - 1 \geq g' - 1 \geq m_i(g_i - 1) \geq \nu_i m_i(g - 1).$$

Hence $\nu_i m_i \leq 2$ and so the only possibilities are

$$\nu_i = m_i = 1, \quad \nu_i = 1, m_i = 2, \quad \text{and} \quad \nu_i = 2, m_i = 1.$$

In the former case by (11) there can be at most two such components, while in the latter two cases at most one. We have $n = \sum_i \nu_i m_i$, the sum over all components C_i of C_0 of positive genus. It follows that $1 \leq n \leq 2$. Then Corollary 2.9 yields $g' \geq g_{d,1} - 3$, since $g_{d,1} - 3 < g_{d,2} - 9$ for $d \geq 6$. Thus we must have

$$\frac{(d-1)(d-2)}{2} - 3 = g_{d,1} - 3 \leq g' \leq 3(g-1) \leq \frac{d(d-5)}{2}, \quad (14)$$

the last inequality coming from (6) for $d \geq 6$. But (14) gives a contradiction. \square

3. BOUNDING GEOMETRIC GENERA OF DIVISORS ON GENERAL 3-FOLDS IN \mathbb{P}^4

A simple way of constructing higher dimensional scrolls consists in starting with the trivial \mathbb{P}^1 -bundle $\pi : S = E \times \mathbb{P}^1 \rightarrow E$ over a smooth projective variety $E \subseteq \mathbb{P}^m$ of degree d and dimension n . Let $\text{Seg}_{a,b}$ denote the image of $\mathbb{P}^a \times \mathbb{P}^b$ via the *Segre embedding*. Then

$$S \hookrightarrow \mathbb{P}^m \times \mathbb{P}^1 \xrightarrow{\cong} \text{Seg}_{m,1} \hookrightarrow \mathbb{P}^{2m+1}$$

yields an embedding of S as a smooth scroll of dimension $n+1$ and degree $(n+1)d$ in \mathbb{P}^{2m+1} . A general linear projection of S to \mathbb{P}^{n+2} gives a hypersurface scroll $\Sigma \subseteq \mathbb{P}^{n+2}$ of degree $(n+1)d$.

Consider, for instance, a surface E_d in \mathbb{P}^3 of degree d , which we suppose to be very general. The above construction gives

$$S_d := E_d \times \mathbb{P}^1 \hookrightarrow \text{Seg}_{3,1} \hookrightarrow \mathbb{P}^7,$$

and S_d is a threefold of degree $3d$ in \mathbb{P}^7 . A general linear projection of S_d to \mathbb{P}^4 yields a threefold scroll Σ_d of degree $3d$ in \mathbb{P}^4 . It is swept out by a two-dimensional family of $(3d-3)$ -secant lines to the double surface Δ_Σ (see Lemma 1.1).

The following version of the Albanese inequality follows immediately from the Semistable Reduction Theorem [34, §1] and the Geometric Genus Criterion (see formula (1) on p. 119 in [34, §6] or, in the surface case, formula (8) in [29, Ch. 5, §5]).

Lemma 3.1. *Let X be a flat limit of a one-parameter family of smooth, irreducible, projective varieties of geometric genus ρ . Let X_i be irreducible components of X_0 with geometric genera ρ_i , $i = 1, \dots, h$. Then*

$$\rho \geq \sum_{i=1}^h \rho_i.$$

Recall that, in the notation as in (12), the geometric genus $\rho(E_d)$ of a smooth surface E_d of degree d in \mathbb{P}^3 is equal to $\rho(E_d) = \binom{d-1}{3} = N_{d-4} + 1$ (see e.g. [29, Ch. 4, (5.12.2)]). The following lower bound on the geometric genus of the double surface is an analog of (10) in the case of surface scrolls.

Lemma 3.2. *Let $\Sigma_d \subseteq \mathbb{P}^4$ be a threefold scroll of degree $3d$, constructed as before over a very general surface E_d in \mathbb{P}^3 of degree $d \geq 5$ as a base. Then for the geometric genus ρ_d of the double surface Δ_{Σ_d} we have a lower bound*

$$\forall d \geq 5. \quad (15)$$

Proof. Degenerate E_d to $E_{d-1} \cup E_1$, where E_{d-1} and E_1 are general. Then S_d degenerates to the union of S_{d-1} and $S_1 = \text{Seg}_{1,2}$, meeting along the Segre image X of $C \times \mathbb{P}^1$, where $C = E_1 \cap E_{d-1}$. Accordingly, Σ_d degenerates in \mathbb{P}^4 to the union of Σ_{d-1} and Σ_1 , the latter being a hypersurface of degree 3 with a double plane. These threefolds intersect along the general projection Y of X , plus another surface Z . The limit of the double locus Δ_{Σ_d} consists of the union of $\Delta_{\Sigma_{d-1}}$, of the plane Δ_{Σ_1} , and of Z . The ruling determines a dominant rational map $Z \dashrightarrow E_{d-1}$. So there is at least one component Z' of Z with geometric genus $\rho' \geq \rho(E_{d-1})$. Now the first inequality in (15) follows from Lemma 3.1. In particular, $\rho_5 \geq \rho' \geq 1$. By induction for every $d \geq 5$ we obtain

$$\rho_d \geq \rho(E_{d-1}) + \rho_{d-1} \geq \sum_{k=5}^{d-1} \rho(E_k) + \rho_5 \geq \sum_{k=0}^{d-2} \binom{k}{3} = \binom{d-1}{4},$$

as required. \square

It would be interesting to find the precise value of ρ_d .

Theorem 3.3. *Any irreducible surface contained in a very general hypersurface of degree $3d \geq 15$ in \mathbb{P}^4 has geometric genus $\rho \geq \min\{\rho_d, N_{d-4} + 1\}$. In particular, $\rho \geq \rho(E_d)$ if $d \geq 8$.*

Proof. The argument is similar to that in the proof of Proposition 2.1, so we will be brief. We let X_0 be the scroll Σ_d and X be a general hypersurface in \mathbb{P}^4 of degree $3d$. The pencil generated by X_0 and X gives rise as usual to a flat family $f : \mathcal{X} \rightarrow \mathbb{D}$. Suppose that the general fibre of this family contains an irreducible surface Y of geometric genus $\rho < \min\{\rho_d, N_{d-4} + 1\}$. By Lemma 3.2 the limit Y_0 of such a surface in the central fibre does not contain Δ_{Σ_d} . By Lemma 3.1 all of its components have geometric genus $\rho' \leq \rho < \min\{\rho_d, N_{d-4} + 1\} \leq N_{d-4} + 1$. Hence they cannot dominate E_d , which has geometric genus $N_{d-4} + 1$. Thus all components of Y_0 pull-back to S_d to surfaces with zero intersection with the ruling. This yields a contradiction as in the proof of Proposition 2.1. \square

Remark 3.4. G. Xu gave in [49, Theorem 2] a sharp lower bound for the geometric genus of an irreducible divisor on a very general hypersurface of degree $d \geq n + 2$ in \mathbb{P}^n , with $n \geq 4$. Of course Theorem 3.3 above is weaker than Xu's result. However, the method of proof is simple and it may possibly have further applications. Hence it would be interesting to extend Theorem 3.3 to other degrees (non-divisible by 3), as well as to higher dimensions. We wonder also whether in higher dimensions an analog of Proposition 2.6 holds. For instance, one can suggest by analogy that on a very general threefold in \mathbb{P}^4 of degree ≥ 6 , the divisors of geometric genera $\rho' < \rho_d$ form bounded families.

4. DEGENERATION TO SCROLLS AND KOBAYASHI HYPERBOLICITY

4.1. Limiting Brody curves and Hurwitz Theorem. Let V be a subvariety of a hermitian complex manifold. A *Brody curve* in V is a holomorphic map $f : \mathbb{C} \rightarrow V$ satisfying

$$\sup_{z \in \mathbb{C}} \|df(z)\| = \|df(0)\| = 1.$$

By *Brody's reparametrization lemma* ([9]), if V is proper and non-hyperbolic then it contains a Brody curve. Furthermore, from any sequence of Brody curves in V one can extract a subsequence converging to a Brody curve, which is called a *limiting Brody curve*.

Assume there is a proper dominant map $\pi : V \rightarrow C$ onto a smooth projective curve C . If general fibres $D_c = \pi^{-1}(c)$ ($c \in C$) are non-hyperbolic, i.e., contain Brody curves, then every special fibre $D_0 := D_{c_0}$ is non-hyperbolic as well and contains limiting Brody curves. The Hurwitz Theorem imposes constraints on limiting Brody curve with respect to the singularities of D_0 (cf. e.g., [43, §1], [51, Theorem 2.1], and [52, Lemma 1.2]). Let $\Delta_0 = \text{br}(D_0)$ be the set of multi-branch points of D_0 such that locally the branches of D_0 are \mathbb{Q} -Cartier divisors on V , and let Δ be the Zariski closure of Δ_0 . Consider a limit $f : \Omega \rightarrow D_0$ of a sequence of holomorphic maps $f_n : \Omega \rightarrow D_{c_n}$, with $c_n \in C \setminus \{c_0\}$ such that $c_n \rightarrow c_0$, where $\Omega \subseteq \mathbb{C}$ is a connected domain. Hurwitz' Theorem says that, if $f(\Omega) \cap \Delta_0 \neq \emptyset$, then $f(\Omega) \subseteq \Delta$. In particular, if Δ is hyperbolic then any limiting Brody curve in D_0 is contained in $D_0 \setminus \Delta_0$. Hence if both Δ and $D_0 \setminus \Delta_0$ are hyperbolic then all fibres D_c ($c \neq c_0$) close enough to D_0 are hyperbolic as well (cf. [51]).

4.2. A hyperbolicity criterion for hypersurfaces in \mathbb{P}^n . Let X_0, X_∞ be distinct hypersurfaces in \mathbb{P}^n of degree d . Typically, X_∞ will be a general surface of degree d meeting $\text{Sing}(X_0)$ in points, where locally X_0 is a union of two smooth branches intersecting transversally. Consider the associated linear pencil $\{X_t\}_{t \in \mathbb{P}^1}$.

Assume that for a general $t \in \mathbb{P}^1$ the hypersurface X_t is non-hyperbolic. Then there exists a sequence of Brody curves $\varphi_n : \mathbb{C} \rightarrow X_{t_n}$ (with respect to the Fubini-Study metric on \mathbb{P}^n), where $t_n \rightarrow 0$, converging to a limiting (non-constant) Brody curve $\varphi_0 : \mathbb{C} \rightarrow X_0$.

Proposition 4.1. *In the above setting, let $B = X_\infty \cap \overline{\text{br}(X_0)}$. If $\overline{\text{br}(X_0)}$ and $(X_0 \setminus \text{br}(X_0)) \cup B$ are both hyperbolic, then X_t , for $t \neq 0$ close enough to 0, is hyperbolic as well.*

Proof. By Hurwitz' Theorem and the hypotheses, the image of φ cannot be contained in $\overline{\text{br}(X_0)}$, and it can meet $\overline{\text{br}(X_0)}$ only at $(\overline{\text{br}(X_0)} \setminus \text{br}(X_0)) \cup B$. But then it is contained in $(X_0 \setminus \text{br}(X_0)) \cup B$, a contradiction. \square

Remark 4.2. Hurwitz' Theorem cannot be applied at points in $(\overline{\text{br}(X_0)} \setminus \text{br}(X_0)) \cup B$, e.g. at a *pinch point* of $X_0 \subseteq \mathbb{P}^3$, where X_0 is locally analytically isomorphic to the surface $x^2 = yz^2$ in $\mathbb{A}^3 = \mathbb{A}_{\mathbb{C}}^3$ at the origin, or at a base point of the pencil situated on $\text{br}(X_0)$.

Indeed, consider a linear pencil of surfaces given in an affine chart \mathbb{A}^3 of \mathbb{P}^3 as $X_t = \{x^2 - y^2z = t\}$. The origin $\mathbf{0} \in \mathbb{A}^3$ is a pinch point of X_0 and is not a base point of the pencil. Consider also the family of entire curves

$$\varphi_\tau : \mathbb{C} \rightarrow X_t, \quad u \mapsto (u^2 + \tau, u, u^2 + 2\tau), \quad \text{where } t = \tau^2 \in \mathbb{C}.$$

The limiting entire curve $\varphi_0(\mathbb{C}) \subseteq X_0$ passes through the pinch point $\mathbf{0} \in X_0$ and is not contained in the singular locus $\{x = y = 0\} = \text{br}(X_0) \cup \{\mathbf{0}\}$ of X_0 .

Corollary 4.3. *In the same setting as before, consider the normalization $\nu : \bar{X}_0 \rightarrow X_0$. Suppose that $\overline{\text{br}(X_0)}$ is hyperbolic and there is a morphism $\pi : \bar{X}_0 \rightarrow E$ onto a hyperbolic variety E such that for every $x \in E$*

$$\pi^{-1}(x) \setminus \nu^{-1}(\text{br}(X_0) \setminus (X_\infty \cap \text{br}(X_0))) \quad (16)$$

is hyperbolic. Then any hypersurface $X_t \neq X_0$ for t close enough to 0 is hyperbolic. Consequently, a very general hypersurface of degree d in \mathbb{P}^n is algebraically hyperbolic.

Proof. We keep the notation introduced before. Suppose that for $t \in \mathbb{P}^1$ general, X_t is not hyperbolic. Let $\varphi_0 : \mathbb{C} \rightarrow X_0$ be a (non-constant) limiting Brody curve. Since its image cannot be contained in $\text{br}(X_0)$, there is a pullback $\tilde{\varphi}_0 : \mathbb{C} \rightarrow \bar{X}_0$. Since E is hyperbolic, the composition $\pi \circ \tilde{\varphi}_0 : \mathbb{C} \rightarrow E$ is constant. Hence $\tilde{\varphi}_0(\mathbb{C})$ is contained in a fibre $\pi^{-1}(x)$ over a point $x \in E$. Furthermore, it does not meet $\nu^{-1}(\text{br}(X_0) \setminus (X_\infty \cap \text{br}(X_0)))$. Indeed, otherwise $\varphi_0(\mathbb{C})$ would meet $\text{br}(X_0) \setminus (X_\infty \cap \text{br}(X_0))$ and, by Hurwitz' Theorem, it would be contained in $\overline{\text{br}(X_0)}$, which is impossible. Then $\tilde{\varphi}_0(\mathbb{C})$ lies in (16), a contradiction. \square

4.3. Applying scrolls to Kobayashi hyperbolicity.

Proposition 4.4. *We keep the notation as in Subsection 1.1. Let $\Sigma \subseteq \mathbb{P}^n$ be a hypersurface scroll with ordinary singularities satisfying conditions (C1)-(C5). Suppose that:*

- (i) *the base E of Σ and its double locus Δ_Σ are both hyperbolic;*
- (ii) *for a general hypersurface X in \mathbb{P}^n of degree $d = \deg(\Sigma)$, every ruling F of Σ meets $\text{br}(\Sigma)$ in at least three distinct points off $X \cap F$.*

Then a general hypersurface of degree d in \mathbb{P}^n is hyperbolic.

Proof. The assertion follows by applying Corollary 4.3 with $X_\infty = X$, $X_0 = \Sigma$, and $\overline{\text{br}(X_0)} = \Delta_\Sigma$. \square

Consider a general sextic scroll of genus 2 as introduced in Example 1.9 and a general septic scroll, also of genus 2, with ordinary singularities in \mathbb{P}^3 . The latter scroll exists according to Theorem 1.5 and Remark 1.6, (ii).

Lemma 4.5. *For a general scroll $\Sigma \subseteq \mathbb{P}^3$ of genus 2 and degree either $d = 6$ or $d = 7$, the following hold:*

- (i) *the projection $\pi : \Delta_S \rightarrow E$ has only simple ramifications; in particular Δ_S meets every ruling in at least three distinct points;*
- (ii) *no pair of pinch points on S sit on the same ruling;*
- (iii) *the rulings passing through the pinch points on S are not tangent to Δ_S .*

Proof. We first treat the case $d = 6$.

The conditions (i)–(iii) are open in $\mathcal{H} = \overline{\mathcal{H}_{6,2}}$. So it suffices to show that there is a surface in \mathcal{H} satisfying these conditions. The reducible surface Σ_0 in Example 1.9 could be used for this, once we know that the analogues of (i)–(iii) hold for a general elliptic quintic scroll. This is in fact the case, but we do not dwell on this here. We use instead a different degeneration of a general sextic scroll of genus 2. We keep the notation introduced in Example 1.9.

A smooth quadric \tilde{Q} in \mathbb{P}^4 can be viewed as a hyperplane section of the Grassmanian $\text{Gr}(1, 3)$ under the Plücker embedding of $\text{Gr}(1, 3)$ in \mathbb{P}^5 . There exists a curve E_0 of degree 6 and arithmetic genus 2 on \tilde{Q} ,

which consists of three conics $\Gamma_0, \Gamma_1, \Gamma_2$ such that Γ_1, Γ_2 are disjoint and intersect both Γ_0 transversally at two points. Indeed, it is enough to take two general hyperplanes H_1, H_2 in \mathbb{P}^4 meeting in a plane L_0 and two other general planes $L_i \subseteq H_i$, $i = 1, 2$, and let $\Gamma_i = L_i \cap \tilde{Q}$, $i = 0, 1, 2$.

The surface $\Sigma_0 \subseteq \mathbb{P}^3$, which corresponds to the curve E_0 , is the union of the three quadrics Q_0, Q_1, Q_2 , corresponding to $\Gamma_0, \Gamma_1, \Gamma_2$, respectively. We may suppose that these quadrics are smooth. The surface Σ_0 belongs to \mathcal{H} . One has $Q_0 \cap Q_i = F_{ij} \cup G_{ij}$, $i, j = 1, 2$, where the lines F_{ij} 's correspond to the intersection points of Γ_0 with Γ_i and belong to the same rulings of Q_0 and Q_i , and G_{ij} are lines of the other rulings of Q_0 and Q_i . Furthermore $Q_1 \cap Q_2 = \varrho$ is a smooth quartic curve of genus 1. By taking Q_0, Q_1, Q_2 sufficiently general, we may suppose that the lines F_{ij}, G_{ij} are general in their rulings and ϱ is also general. We denote by $p_{ij;hk}$ the intersection of F_{ij} with G_{hk} , where $i, j, h, k \in \{1, 2\}$. We note that ϱ meets Q_0 at the eight points $p_{ij;3-i,h}$, with $i, j, h \in \{1, 2\}$.

Regard now Σ_0 as a limit of a general sextic scroll Σ of genus 2. The points of $\Gamma_0 \cap \Gamma_i$, $i = 1, 2$, are smoothed when deforming E_0 to E , hence also the lines F_{ij} are. Therefore the limit of the smooth double curve $C = \Delta_\Sigma$ is the curve

$$C_0 = \Delta_{\Sigma_0} = \varrho \cup \bigcup_{i,j=1,2} G_{ij}$$

of degree 8 and arithmetic genus 5. The limit on Σ_0 of the ruling on Σ is the union of the rulings of Q_0, Q_1, Q_2 containing the lines F_{ij} , $1 \leq i \leq j$. By (7) there are 16 pinch points on Σ . Similarly as in Example 1.7, each of the eight points $p_{ij;ih}$, $i, j, h = 1, 2$ (not lying on ϱ) is the limit of two pinch points of Σ . We call them *limit pinch points*.

The smooth normalization S of Σ specializes to a partial normalization S_0 of Σ_0 , ruled over the same nodal base curve E_0 . The singular surface S_0 consists of three irreducible quadric surfaces $\tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2$ glued together along the common rulings \tilde{F}_{ij} in the same way as before.

The limit $\tilde{C}_0 = \Delta_{S_0}$ of $\tilde{C} = \Delta_S$ is a nodal curve of arithmetic genus 17. It maps to E_0 with degree 4, and consists of ten components:

- two copies $\varrho_i \subseteq \tilde{Q}_i$ of ϱ , each is mapped with degree 2 to Γ_i , $i = 1, 2$;
- two copies $G_{i;hk} \subseteq \tilde{Q}_i$ of G_{hk} , with $h, k = 1, 2$ and $i \in \{0, h\}$, eight curves in total. The curves $G_{0;hk}$ and $G_{h;hk}$ are glued at two points $p_{h1;hk}$ and $p_{h2;hk}$. Each of them is also glued to ϱ_h at two points, $h = 1, 2$. Hence the curves ϱ_1 and ϱ_2 meet in the eight points $p_{ij;3-i,h}$, with $i, j, h = 1, 2$. The four disjoint curves $G_{0;hk}$ on \tilde{Q}_0 are all mapped isomorphically to Γ_0 , whereas for $h = 1, 2$ the two disjoint curves $G_{h;hk}$ on \tilde{Q}_h ($k = 1, 2$) are mapped isomorphically to Γ_h .

Therefore, the limit of the 24 ramification points of the projection $\pi : \tilde{C} \rightarrow E$ are:

- (a) the ramification points of the degree 2 covers $\varrho_i \rightarrow \Gamma_i$, $i = 1, 2$, in total 8 such points;
- (b) the connecting nodes of ϱ_h with $G_{h;hk}$, $k, h = 1, 2$, in total 8 distinct such points, each counted with multiplicity two.

We call these the *limit ramification points*.

Part (i) follows from this description, our generality assumption, and the observation that every limit ramification point of type (b) smooths to two ramification points on \tilde{C} lying on different rulings.

As for (ii), the ruling F_{ij} through $p_{ij;ih}$ misses all limit pinch points other than $p_{ij;i,3-h}$. Consider a partial deformation of Σ_0 to the union of a general elliptic quartic scroll Σ'_0 and a quadric Q'_1 containing two general rulings. This corresponds to a partial smoothing of E_0 to the union of an elliptic quartic curve E' , obtained by smoothing $\Gamma_0 + \Gamma_2$, plus a conic Γ'_1 (specializing to Γ_1) meeting E' transversally at two points. In this way Σ'_0 has two double lines R_1, R_2 which respectively specialize to G_{21} and G_{22} . For a fixed index $i \in \{0, 1\}$, the two limit pinch points $p_{2j;2i}$, $j = 1, 2$, deform to four pinch points of Σ'_0 on R_i , and, as we saw in Example 1.7, they are general points on R_1, R_2 and are never pairwise on a ruling.

For the proof of (iii) note that, by generality assumptions, the rulings through the limit ramification points of type (a) do not contain any of the limit pinch points. In contrast, the rulings through limit ramification points of type (b) do contain limit pinch points. However, the same proof as for (ii) and generality assumptions imply that, in a general deformation of Σ_0 to Σ , this is no longer the case.

The case $d = 7$ is similar, hence we will be as brief as possible. The closure of $\mathcal{H}_{7,2}$ contains points corresponding to a surface $\Sigma_0 = \Sigma' \cup P$, where Σ'_0 is a general sextic scroll of genus 2 and P is a general plane containing a general ruling F . The intersection of P with Σ' consists of F plus a plane quintic curve D of genus 2, which has four nodes p_i , $i = 1, \dots, 4$. The intersection of $C' = \Delta_{\Sigma'}$ with P consists of the points p_i , $i = 1, \dots, 4$, and four more points $q_i \in F$, $i = 1, \dots, 4$. The intersection of D with F consists of the points q_i , $i = 1, \dots, 4$, and of a further point q which is smooth on Σ' , so that P is tangent to Σ' at q .

The surface Σ_0 is the limit of a general scroll Σ of degree 7 and genus 2. If E is the base of Σ regarded as a curve in $\text{Gr}(1, 3)$, this corresponds to E degenerating to E_0 , which is the union of a general sextic E' of genus 2 and a line L meeting E' transversally at one point f , which corresponds to F . The limit of the ruling of Σ is the ruling of Σ' plus the pencil in P , corresponding to L , with center a general point of F .

The limit of $C = \Delta_{\Sigma}$ is the curve $C_0 = C' \cup D$ of degree 13. The points p_i , $i = 1, \dots, 4$, are limits of the four triple points of C . The geometric genus of a partial smoothing of C_0 at the points q_i , $i = 1, \dots, 4$, is 10. All this agrees with (5), (6), and (7).

The usual analysis shows that the limit of the 18 pinch points of Σ (see (8)) are the 16 pinch points of Σ' plus the point q counted with multiplicity 2.

The limit \tilde{C}_0 of $\tilde{C} = \Delta_S$ maps with degree five to the curve $E_0 = E' \cup L$. It consists of:

- a copy \tilde{C}' of $\Delta_{S'}$ which maps to E' with degree four;
- a copy of the normalization \bar{D} of D , which maps isomorphically to E' and meets \tilde{C}' transversally at four points;
- a copy of D which maps to L with multiplicity five via the projection induced by the ruling on P , and meets \tilde{C}' transversally at four points.

Hence the limit of the 44 ramification points of the projection $\pi : \tilde{C} \rightarrow E$ are

- the 24 ramification points of the map $\tilde{C}' \rightarrow E'$;
- the 12 ramification points of the map $D \rightarrow L$;
- the 4 connecting nodes of \tilde{C}' with \bar{D} , each counted with multiplicity two.

With this in mind the proof proceeds similarly to the case $d = 6$. The details can be left to the reader. \square

Theorem 4.6. *For every $d \geq 6$ there exists a hyperbolic surface in \mathbb{P}^3 of degree d . Consequently, a very general surface in \mathbb{P}^3 of degree $d \geq 6$ is algebraically hyperbolic.*

Proof. For $d = 6, 7$ this follows from Corollary 4.3 and Lemma 4.5. For $d \geq 8$ one can consider e.g. a general deformation of the union of two general cones in \mathbb{P}^3 of degrees d_1, d_2 , where $d_1 + d_2 = d$ and $d_i \geq 4$ (see [44]). \square

Remark 4.7. Consider the union $X_0 = X_1 \cup X_2$ of projective cones with distinct vertices in \mathbb{P}^4 over two smooth hyperbolic surfaces in \mathbb{P}^3 . According to [44], X_0 can be deformed to a smooth hyperbolic threefold in \mathbb{P}^4 of degree $\deg(X_1) + \deg(X_2)$. Thus there exist hyperbolic threefolds in \mathbb{P}^4 of any given degree $d \geq 12$. Consequently, a very general threefold in \mathbb{P}^4 of degree $d \geq 12$ is algebraically hyperbolic.

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